A Critical Study of the Coaxial Transmission Line Utilizing Conductors of Both Circular and Square Cross Section

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***Abstract*—By means of various conformal transformations we calculate the characteristic impedance of the coaxial line consisting of both square and circular cylinders. We also employ a complex potential function to match the boundary condition on one conductor completely and on either 12 points or 8 points on the other conductor to obtain exceptionally accurate results. Our present results are believed to be considerably better than our previous results quoted by Gunston [2].**

I. Introduction

Riblet [1] has made an ingenious use of Bowman's work to determine the characteristic impedance of the coaxial transmission line utilizing conductors of both circular and square cross section, as shown in Fig. 1. The rectangular region in the w-plane, bounded by $0ABC$ is mapped conformally into the trapezoidal region in the z-plane of Fig. 1, bounded by $0ABC$. Riblet has stated that an equipotential line GH in the w-plane transforms into an equipotential in the z-plane, which is circular within ± 0.2 percent for impedance values in the vicinity of 25 Ω and which is circular within two parts in 10^7 for those in the vicinity of 100 Ω. It is the purpose of this paper to make a fuller use of the transformation of Fig. 1, to improve the accuracy of the calculation of the impedances of lines, having inner circular and outer square conductors, improving also some of our previous results [2]–[4]. We also give exact bounds of the mid equipotential in z-plane, to show that it is far from being circular.

II. Theory

We can find the exact characteristic impedance Z_0 of the coaxial line consisting of two square cylinders with diagonals at 45° as shown in the z-plane in Fig. 1 where we have shown one quarter of the coaxial system under study. Then $Z_0/2$ is the characteristic impedance of both the lines, one conductor of which is the closed cylinder with HGF as one quarter of it and the other conductor is either an outer square or an inner square cylinder. If HGF is indeed a circular arc, then we have achieved our goal. We find that is not so.

Instead of utilizing the original transformation of Bowman [1], we take the symmetric region $ABCDEA$, in the

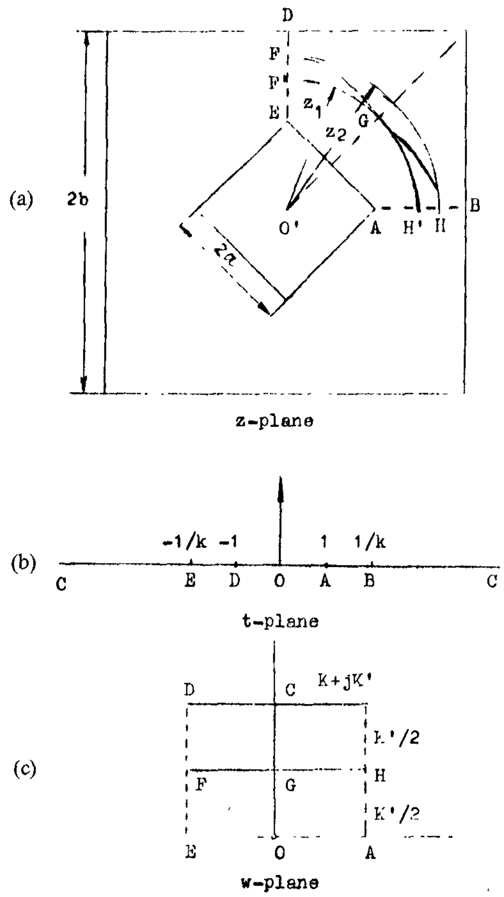

Fig. 1.

z-plane and by the following transformations we map it into the corresponding region $ABCDE$ of the w-plane:

$$z = A\int_0^t \frac{dt}{(t^2-1)^{1/4}(t^2-1/k^2)^{1/2}} + ae^{j\pi/4} \tag{1a}$$

$$t = \mathrm{sn}(w,k) \tag{1b}$$

where $\mathrm{sn}(w,k)$ is the Jacobian Elliptic function of modulus k, so that

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}\,\frac{K'(k)}{8K(k)}. \tag{2}$$

While k is related to the sizes of the squares in the z-plane of Fig. 1 by the fact that $t=1$, $z=\sqrt{2}\,a$, and $t=1/k$, $z=b$,

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TABLE I
IMPEDANCE Z_0: (SEE FIG. 1(a))

Z_0 (Ohms)	20	30	40	50	60	80	100	120	140	160	180	200	220	240
a/b	.638732	.547498	.46400	.388874	.323103	.23232	.16823	.121257	8.77273e-2	.063368	4.57023e-2	3.26864e-2	2.38845e-2	1.70101e-2

and, therefore, the constant

$$A = ja \Big/ \int_0^1 \frac{dt}{(1-t^2)^{1/4}\left(\dfrac{1}{k^2}-t^2\right)^{1/2}} \tag{3}$$

is pure imaginary and k is to be obtained by the following relation:

$$\frac{b}{a}-\sqrt{2} = \frac{\displaystyle\int_1^{1/k} \frac{dt}{(t^2-1)^{1/4}\left(\dfrac{1}{k^2}-t^2\right)^{1/2}}}{\displaystyle\int_0^1 \frac{dt}{(1-t^2)^{1/4}\left(\dfrac{1}{k^2}-t^2\right)^{1/2}}} \tag{4}$$

once the ratio b/a is prescribed.

Equation (4) fixes the relationship between b/a and k, then by (2), that between Z_0 and the ratio b/a. By (4) and (3) we calculate the impedance Z_0 versus b/a (see Table I).

Now we study the mid-equipotinal line HGF in the z-plane corresponding to the mid-equipotential line HGF in the uniform field of the w-plane in Fig. 1. For the line HGF we have

$$w = u + j\frac{K'}{2}$$

putting this value of w into (1b), we find

$$t = \frac{1}{\sqrt{k}}\,\frac{(1+k)\,\mathrm{sn}(u,k) + j\,\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}. \tag{5}$$

(5) is to be inserted in the upper limit of the integral defining z, to determine the locus HGF in the z-plane. To examine if HGF is nearly circular, all we have to do is to find the maximum and minimum values of $|z|$ along the curve HGF. So we solve for z in the following equation:

$$\frac{d}{du}|z|^2 = 0, \qquad \text{i.e., } x\frac{dx}{du} + y\frac{dy}{du} = 0$$

that is,

$$\mathrm{Re}\left(z^*\frac{dz}{du}\right) = 0$$

where Re denotes "the real part of" and $z^* = x - jy$; we can finally have

$$\mathrm{Re}\left(z^*\frac{dz}{dt}\,\frac{dt}{du}\right) = 0 \tag{6}$$

to obtain $|z|_{max}$ and $|z|_{min}$ along GHF in the z-plane.

Now, from (1a) we have

$$\frac{dz}{dt} = A\frac{1}{(t^2-1)^{1/4}(t^2-1/k^2)^{1/2}} \tag{7a}$$

while from (5) we find that

$$\frac{dt}{du} = \frac{(1+k)\left[1-k\,\mathrm{sn}^2(u,k)\right]}{\sqrt{k}} \cdot \frac{\mathrm{cn}(u,k)\,\mathrm{dn}(u,k) + j(1+k)\,\mathrm{sn}(u,k)}{\left[1+k\,\mathrm{sn}(u,k)\right]^2}. \tag{7b}$$

It can readily be seen from (5) that at $u=0$, $t = j/\sqrt{k}$. Therefore, from (7a)

$$\frac{dz}{dt} = |A|\frac{k}{\left(1+\dfrac{1}{k}\right)^{3/4}}e^{-j\pi/4} \tag{8a}$$

and also from Fig. 1 it is seen that along $0C$

$$z^* = |z|e^{-j\pi/4} \tag{8b}$$

and finally at $u=0$

$$\frac{dt}{du} = \frac{1+k}{\sqrt{k}}. \tag{8c}$$

Putting (8a–c) into (6), (6) holds, because the product of (8a–c) is pure imaginary. Similarly, at $u=K$, $t = 1/\sqrt{k}$, z^* is real at the point H of the t-plane of Fig. 1 and from (7a), dz/dt is real, but from (7b), dt/du is purely imaginary so (6) again holds. Thus, we conclude that $|z_H|$ and $|z_G|$ are extreme values, which are given by (1a) to be

$$z_1 = |z|_{min} = \left| A\int_0^{j/\sqrt{k}} \frac{dt}{(t^2-1)^{1/4}(t^2-1/k^2)^{1/2}} + ae^{j\pi/4}\right|$$
$$= a + |A|\int_0^{1/\sqrt{k}} \frac{dy}{(y^2+1)^{1/4}\left(y^2+\dfrac{1}{k^2}\right)^{1/2}} \tag{9a}$$

and

$$z_2 = |z|_{max} = \sqrt{2}\,a + |A|\int_0^{1/\sqrt{k}} \frac{dt}{(t^2-1)^{1/4}\left(\dfrac{1}{k^2}-t^2\right)^{1/2}} \tag{9b}$$

where A is given by (3).

TABLE II

a/b	Z_0, ohms	z_1/b	z_2/b	$\eta = \frac{z_2-z_1}{z_2+z_1} \times 100$
3.23075e-2	199.019	7.30082e-2	.202962	23.54
3.9568e-2	186.887	8.88595e-2	.224616	21.65
4.84624e-2	175.238	.108006	.248586	19.71
5.3556e-2	163.485	.131057	.275122	17.73
7.26988e-2	151.554	.158691	.30451	15.74
8.90464e-2	139.505	.191645	.33706	13.75
.10117	131.889	.215444	.359339	12.52
.109082	127.39	.230683	.37317	11.80
.122677	120.363	.256325	.395831	10.70
.133662	115.228	.27655	.41326	9.91
.148249	109022	.302723	.435361	8.99
.163879	103.014	.329914	.457909	8.12
.18331	96.2913	.362518	.484567	7.20
.201203	90.7063	.391341	.507933	6.48
.225389	83.8912	.428566	.538051	5.66
.247822	78.1898	.461405	.564693	5.03
.274415	72.0572	.49825	.594926	4.40
.307679	65.1594	.541371	.631023	3.82
.333788	60.2354	.573064	.658246	3.46
.390835	50.6432	.636385	.715037	2.91
.411289	47.5185	.657313	.734677	2.78
.432222	44.4614	.677867	.754458	2.67
a/b	Z_0, ohms	z_1/b	z_2/b	$\eta = \frac{z_2-z_1}{z_2+z_1} \times 100$
.454719	41.3152	.699057	.775406	2.59
.479771	37.9594	.721654	.79841	2.52
.509363	34.1628	.747147	.825225	2.48
.549007	29.2943	.779573	.860693	2.47
.571943	26.5541	.797597	.881042	2.49
.587327	24.7317	.809443	.894641	2.50
.609133	22.1471	.825968	.913868	2.53
.626523	20.0591	.838976	.929173	2.55
.637133	18.7586	.846862	.938496	2.57
.65510	16.4685	.860193	.954282	2.59

Once the ratio a/b is prescribed, Z_0 is fixed, then $Z_0/2$ is the upper bound to the characteristic impedance of the line formed by an inner circular conductor of radius z_2 and the outer conductor of the square of side b, and this $Z_0/2$ is also the upper bound to the characteristic impedance of the line formed by an inner conductor of the square of side a and an outer circular conductor of radius z_1. Making use of this upper bound and the lower bound obtained previously, to be described below, we can improve the accuracy of the calculated characteristic impedance of the coaxial line utilizing conductors of both circular and square cross section.

We may employ two such equipotentials GH and $G'H'$ shown in Fig. 2 to be the upper and the lower bounds to the characteristic impedances of the lines utilizing a circular cylinder of radius R and a square cylinder of sides a or b, but these lower and upper bounds are found to be too far apart to give accurate results.

In Table II, we tabulate the values of Z_0, z_1, and z_2 against a/b and the values of the percent deviation of

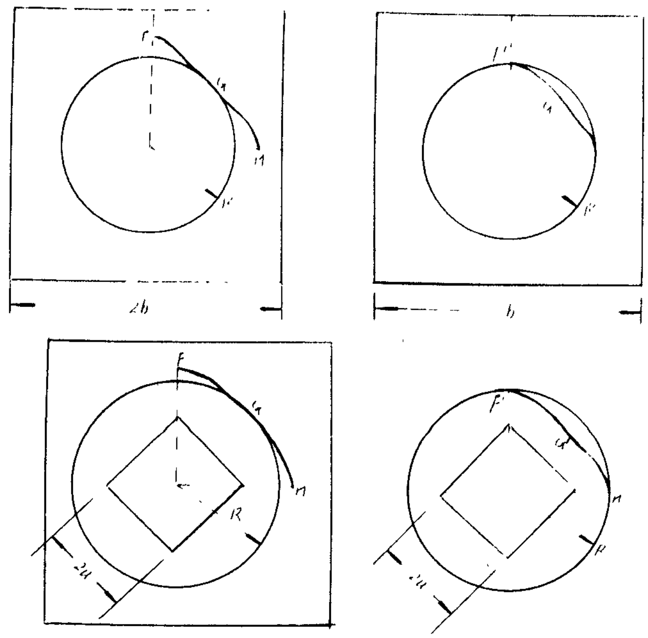


Fig. 2.

$$\frac{b_1}{b} = \frac{K_1'(k_1)}{K_1(k_1)}$$

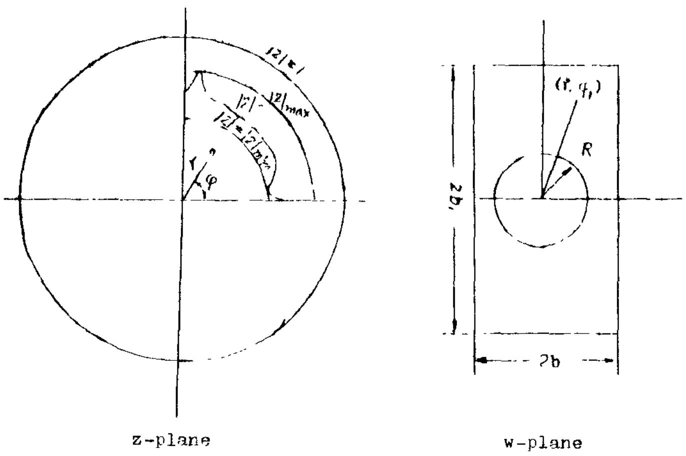


Fig. 3.

$\Delta z = (z_1 - z)/2$ from the mean value of z_1 and z_2

$$\eta = \frac{z_2 - z_1}{z_2 + z_1} \times 100$$

when k in Fig. (1b) is increasing from $k = 0.005$ to $k = 0.999$. So we can see that the mid-equipotenial GH is far from being circular when $Z_0/2$ increases from 8.234 Ω to 99.51 Ω.

A. Improved Results

Let us call Z_{01} and Z_{02} the characteristic impedance, respectively, of the line with outer square and inner circular cylinders and that with outer circular and inner square cylinders.

B. The Impedance Z_{01}

We now take the following transformation [2]–[4] to map the interior of a rectangular of w-plane into that of a unit circle of the z-plane and to map a circle of radius R of the w-plane into a closed curve of the z-plane lying between the circles $z = |z|_{\max}$ and $z = |z|_{\min}$, Fig. 3

$$z = \frac{\mathrm{sn}\left(\frac{Kw}{2b}, k_1\right)\mathrm{dn}\left(\frac{kw}{2b}, k_1\right)}{\mathrm{cn}\left(\frac{Kw}{2b}, k_1\right)} = \sqrt{\frac{1-\mathrm{cn}\left(\frac{Kw}{b}, k_1\right)}{1+\mathrm{cn}\left(\frac{Kw}{b}, k_1\right)}}. \tag{10}$$

In our case, $b = b_1$ and $k_1 = 1/\sqrt{2}$, so we have

$$\text{Upper bound: } (Z_{01})_{\max} = 59.952 \ln|z|_{\max} \Omega$$
$$\text{Lower bound: } (Z_{01})_{\min} = 59.952 \ln|z|_{\min} \Omega \tag{11}$$

where

$$|z|^2_{\max} = \frac{l + \mathrm{cn}^2\left\{\frac{R}{2b}K(0.707107), 1/\sqrt{2}\right\}}{l - \mathrm{cn}^2\left\{\frac{R}{2b}K(0.707107), \frac{1}{\sqrt{2}}\right\}} \tag{11a}$$

and

$$|z|^2_{\min} = \frac{l + \mathrm{cn}\left\{\frac{R}{b}K(0.707107), 1/\sqrt{2}\right\}}{l - \mathrm{cn}\left\{\frac{R}{b}K(0.707107), 1/\sqrt{2}\right\}} \tag{11b}$$

where K is the complete elliptic integral of the first kind.

We tabulate the values Z_{01} from (11), and also the arithmetic and geometric mean values

$$(Z_{01})_{\text{a m.}} = \frac{(Z_{01})_{\max} + (Z_{01})_{\min}}{2} \tag{12}$$

$$(Z_{01})_{\text{G.m.}} = \{(Z_{01})_{\max}(Z_{01})_{\min}\}^{1/2} \tag{13}$$

in Table III. Also, we tabulate the following match point impedance:

$$(Z_{01})_{\text{match}} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}A_0^{-1} = 59.952 A_0^{-1} \tag{14}$$

where A_0 is the first coefficient in the truncated series of circular harmonics for the complex potential in the z-plane [3]. For our case of square electrodes, so that an addition of 90° to φ of Fig. 2 does not alter the complex potential, which is then of the form

$$\Phi = A_0 \ln_z + A_4\left(z^4 - \frac{1}{z^4}\right) + A_8\left(z^8 - \frac{1}{z^8}\right) \tag{15a}$$

so that

$$\begin{aligned}\Phi = U + jV = {} & A_0 \ln r + A_4\left(r^4 - \frac{1}{r^4}\right)\cos 4\phi \\ & + A_8\left(r^8 - \frac{1}{r^8}\right)\cos 8\varphi \\ & + j\left\{A_0\varphi + A_4\left(r^4 + \frac{1}{r^4}\right)\sin 4\varphi\right. \\ & \left. + A_8\left(r^8 + \frac{1}{r^8}\right)\sin 8\varphi.\right\}\end{aligned} \tag{15b}$$

Therefore, $U = 0$ at the outer square electrode and fits the boundary condition at 12 points on the inner electrode if we take

$$U = 1 \text{ at } \varphi = 0, r = r_1 = |z|_{\min};\ \varphi = \pi/8, r = r_2, \text{ and } \varphi = \frac{\pi}{4}, r = r_3 = |z|_{\max}$$

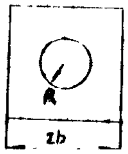
where

$$r_2^2 = \frac{1 - \mathrm{cn}\left(\frac{1.854075}{b/R}\cos\frac{\pi}{8}, \frac{1}{\sqrt{2}}\right)\mathrm{cn}\left(\frac{1.854075}{b/R}\sin\frac{\pi}{8}, 1/\sqrt{2}\right)}{1 + \mathrm{cn}\left(\frac{1.854075}{b/R}\cos\frac{\pi}{8}, \frac{1}{\sqrt{2}}\right)\mathrm{cn}\left(\frac{1.54075}{b/R}\sin\frac{\pi}{8}, \frac{1}{\sqrt{2}}\right)} \tag{16}$$

and we can set up the system of linear equations to solve

TABLE III
Z_{01}

Z_{01}:



$\frac{R}{b}$	0.0470973	0.142010	0.239085	0.3399780	0.44515	0.557536	0.676602	0.802495	0.93352
$(Z_{01})_{min}$	187.728	121.558	90.3150	69.1987	52.8401	39.1389	27.0309	15.8778	5.23845
$(Z_{01})_{ma.}$	187.747	121.576	90.3716	69.3194	53.2166	39.9990	28.9102	19.5630	11.9807
$(Z_{01})_{a.m.}$	187.737	121.567	90.34330	69.2590	53.0284	39.5690	27.9705	17.7204	8.60960
$(Z_{01})_{g.m.}$	187.737	121.567	90.3433	69.2590	53.0280	39.5666	27.9548	17.62440	7.92216
Maximum o/o error of $(Z_{01})_{a.m.}$	0.0050765	0.007264	0.0313857	0.087253	0.356299	1.02224	3.47625	11.6047	64.3539
Maximum o/o error of $(Z_{01})_{G.m.}$	0.0050712	0.007264	0.0313802	0.0872141	0.355665	1.09292	3.41785	10.9998	51.2309
$(Z_{01})_{match}$	187.737	121.5667	90.343	69.2589	53.0260	39.5565	27.9088	17.4483	7.17168
Maximum o/o error $(Z_{01})_{match}$	0.0050699	0.0072592	0.0313466	0.086901	0.351817	1.067045	3.24793	9.89073	36.9046

for A_0

$$A_0 \ln r_1 + \left(r_1^4 - \frac{1}{r_1^4}\right) A_4 + \left(r_1^8 - \frac{1}{r_1^8}\right) A_8 = 1$$

$$A_0 \ln r_2 \cdot - \left(r_2^8 - \frac{1}{r_2^8}\right) A_8 = 1$$

$$A_0 \ln r_3 - \left(r_3^4 - \frac{1}{r_3^4}\right) + \left(r_3^8 - \frac{1}{r_3^8}\right) A_8 = 1. \quad (17a)$$

It follows then

$$A_0 = \frac{R_{11}R_{22} + R_{12}R_{31}R_{20} + R_{22}R_{31}R_{10} + R_{32}R_{11}R_{20}}{R_{11}R_{22} + R_{12}R_{31} + R_{22}R_{31} + R_{32}R_{11}} \quad (17b)$$

where

$$R_{10} = \ln r_1 \quad R_{20} = \ln r_2 \quad R_{30} = \ln r_3$$

$$R_{11} = r_1^4 - \frac{1}{r_1^4} \quad R_{12} = r_1^8 - \frac{1}{r_1^8}$$

$$R_{21} = r_2^4 - \frac{1}{r_2^4} \quad R_{22} = r_2^8 - \frac{1}{r_2^8}$$

$$R_{31} = r_3^4 - \frac{1}{r_3^4} \quad R_{32} = r_3^8 - \frac{1}{r_3^8}.$$

For prescribed values of R/b we can calculate the Rij by calculating $r_1 = |z|_{min}$, $r_3 = |z|_{max}$, and r_2 by (11a–b), and (16). $(Z_{01})_{match}$ enters Table III, where we have given also the maximum percent error by the following formula:

$$100\frac{Z_{01} - (Z_{01})_{true}}{(Z_{01})_{true}} \leqslant 100\frac{Z_{01} - (Z_{01})_{min}}{(Z_{01})_{min}}. \quad (18)$$

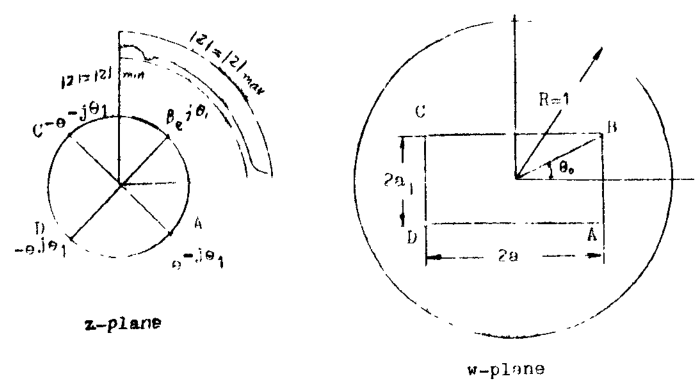


Fig. 4.

C. The Impedance Z_{02}

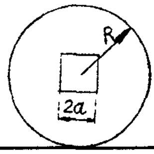
We have shown that the following transformation [4] maps the exterior of the rectangle in the w-plane into that of a unit circle in the z-plane (Fig. 4):

$$w = c\int_{e^{-j\theta_1}}^{z} \frac{dz}{z^2}\left((z^2+1)^2 - 4z^2\cos^2\theta_1\right)^{1/2} + \sqrt{a^2 + a_1^2}\, e^{-j\theta_0} \quad (19)$$

$$\tan\theta_0 = a_1/a = \frac{E(\sin\theta_1) - \cos^2\theta_1 K(\sin\theta_1)}{E(\cos\theta_1) - \sin^2\theta_1 K(\cos\theta_1)} \quad (20)$$

TABLE IV
Z_{02}

Z_{02}:



a/R	.05	.1	.2	.3	.32	.34	.36	.38	.4
$\|z\|_{min}$	16.9456	8.4723	4.234	2.81663	2.6385	2.48097	2.34036	2.21412	2.09997
$\|z\|_{max}$	16.9456	8.47243	4.23827	2.83138	2.65642	2.50242	2.36592	2.2442	2.73508
$(Z_{02})_{min}$	169.665	128.105	86.5195	62.0827	58.166	54.4738	50.9776	47.6532	44.4797
$(Z_{02})_{max}$	169.665	128.106	86.5799	62.3958	58.5718	54.9914	51.6286	48.4621	45.4736
$(Z_{02})_{A.M.}$	169.665	128.106	86.5497	62.2393	58.3689	54.7326	51.3031	48.0577	44.9767
error, o/o	3.48845e-2	.25211	.348796	.475029	.638532	.848735	1.11725	1.45951	1.89385
$(Z_{02})_{G.M.}$	169.665	128.106	86.5501	62.2391	58.3686	54.7321	51.3021	48.0600	44.9739
Max. o/o error	3.48e-2	.25211	.34870	.4750	.6385	.8487	1.116	1.452	1.875
$(Z_{02})_{match}$	169.590	128.048	86.5107	62.2097					44.9399
Max. o/o error	-0.0442	-0.0442	-0.0101	0.2045					1.0346

a/R	.42	.44	.46	.48	.5	.6	.7
$\|z\|_{min}$	1.99611	1.90105	1.81357	1.73261	1.65729	1.34024	1.04465
$\|z\|_{max}$	2.03678	1.94788	1.86716	1.79362	1.72644	1.46398	1.28592
$(Z_{02})_{min}$	41.4385	38.5134	35.6889	32.9513	30.2867	17.5569	2.61876
$(Z_{02})_{max}$	42.6481	39.9722	37.4348	35.026	32.7373	22.8512	15.0764
$(Z_{02})_{A.M.}$	42.0433	39.2428	36.5619	33.9886	31.512	20.204	8.84759
error, o/o	2.44595	3.74817	4.0456	15.0775	237.853		
$(Z_{02})_{G.M.}$	42.0388	39.2360	36.5514	33.9728	31.4882	20.0299	6.28343
Max. o/o error	2.420	3.108	3.960	14.15	140.0		
$(Z_{02})_{match}$					31.395	19.666	4.226
Max. o/o error					3.6594	12.01	61.36

and

$$c = \frac{a/R}{2\left(E(\cos\theta_1) - \sin^2\theta_1 K(\cos\theta_1)\right)}. \tag{21}$$

The circle of radius $R = 1$ in the w-plane is mapped into a closed curve lying between the circles $|z| = |z|_{max}$ and $|z| = |z|_{min}$ in the z-plane. For our case of square and circular electrodes, $a_1 = a$, $\theta_0 = \pi/4$ and

$$\theta_1 = \pi/4, \quad c = \frac{a}{2R}\frac{1}{1.350644 - \frac{1}{2}(1.854075)}. \tag{22}$$

Also $|z|_{max}$ and $|z|_{min}$ are given by the upper limits of the following integrals:

$$1 - \frac{a}{R} = \int_1^{|z|_{max}} \frac{dy}{y^2}\sqrt{y^4+1} \tag{23}$$

$$1 - \sqrt{2}\,a/R = \int_1^{|z|_{min}} \frac{dy}{y^2}\sqrt{y^4-1}. \tag{24}$$

So we calculate $|z|_{max}$ and $|z|_{min}$ from (23) and (24) and then we calculate the bounds and the approximation to the characteristic impedance of the line whose cross section is shown in the w-plane of Fig. 4, by (11)–(14), while in (17a) we set $A_8 = 0$ to simplify the problem at the expense of losing some accuracy. A_0 is then given by

$$A_0 = \frac{\left(|z|_{max}^4 - |z|_{max}^{-4}\right)\ln\frac{1}{|z|_{min}} + \left(|z|_{min}^4 - |z|_{min}^{-4}\right)\ln\frac{1}{|z|_{max}}}{|z|_{max}^4 - |z|_{max}^{-4} + |z|_{min}^4 - |z|_{min}^{-4}}.$$

In Table IV, we show the results of the calculations.

III. Discussion and Conclusion

By comparing the values of $(Z_{01})_{max}$ in Tables II and III, we find that we may pick the smaller upper bound, and together with the lower bound we can improve the accuracy of the calculated value of Z_{01}. For instance, for $z_1/b = 0.802495$ and 0.933521 we find $(Z_{01})_{max}$ to be $1/2(34.7412) = 17.3706\ \Omega$ and $1/2(19.6012) = 9.8006\ \Omega$ in Table II, smaller than those in Table III at the same R/b ratio. So we can calculate the new $(Z_{01})_{G.M.}$ values to be

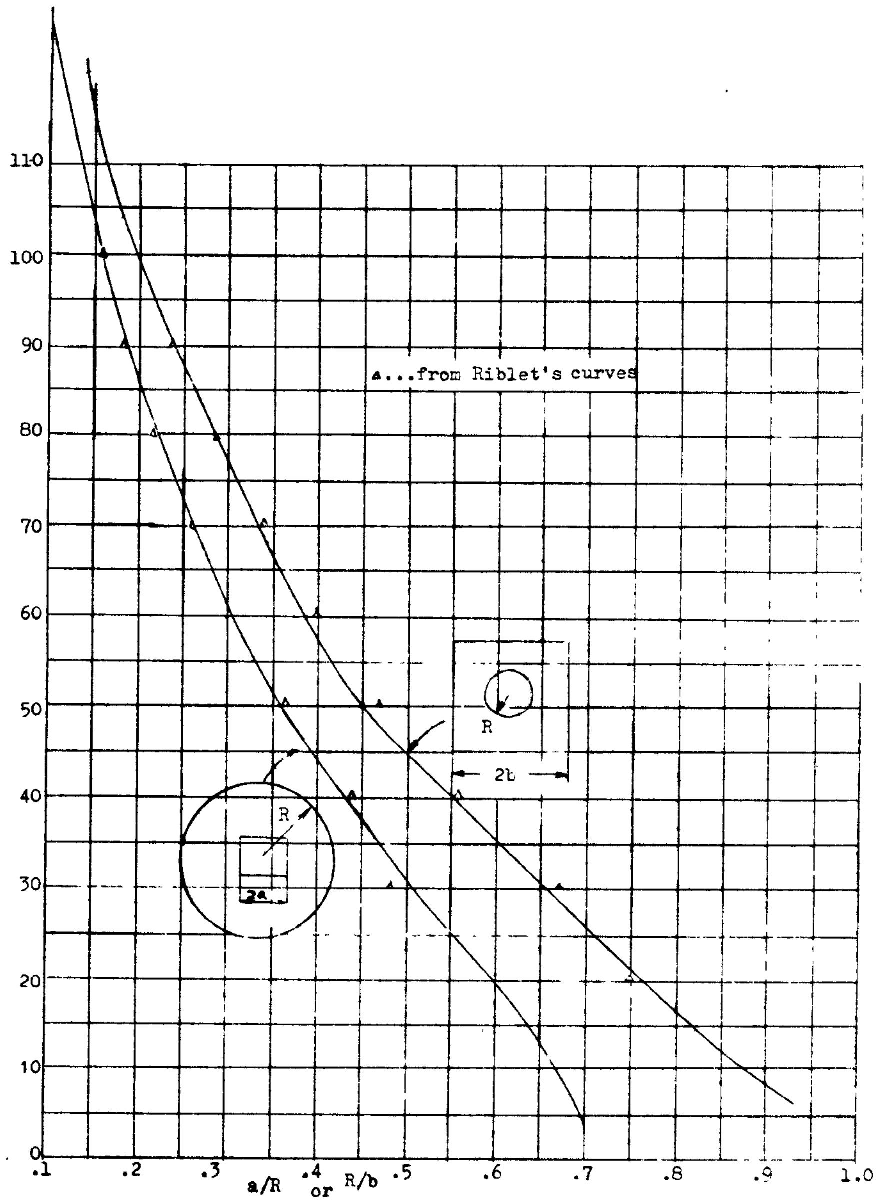


Fig. 5.

TABLE V
RECOMMENDED VALUES OF Z_{01}

R/b	.04710	.14201	.23909	.33978	.44552	.55754	.67660	.80520	.93352
Z_{o1}	187.737	121.5667	90.5667	69.2589	53.0260	39.5565	27.9088	16.7758	7.1652
Max o/o error	.0051	.0073	.0314	.0869	.3518	1.067	3.248	5.66	36.78

16.7758 and 7.1652 with maximum 0/0 error of 5.66 and 36.78. However, we can achieve no improvement in the calculated values of Z_{02}. So we would recommend to adopt the last two lines of Table IV for the values of Z_{02} and the values of Table V as the values of Z_{01}, for impedances of the coaxial line of square and circular cylinders. These are plotted in curves in Fig. 5 together with data from curves of Riblet's paper and from curves of Laura and Luisoni.

For better bounds than we have obtained analytically here, we have to resort to numerical calculations [6], [7]. Riblet's work has been supported by the approximate treatments [5]. M.A.R. Gunston has made a good comparison of works by various authors on this and other related transmission line problems [2]. The results reported in this paper are considerably better than those of our previous works quoted by Gunston [2].

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Theory and Application of Coupling Between Curved Transmission Lines

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***Abstract*— An analytical method for deriving the fields, the reflection, and the directivity of two coupled curved transmission lines is described. The fields on both lines are found to be accurately in quadrature. The directivity and reflection are very small. The accuracy of the theoretical results for a 3 dB dielectric line coupler (designed at 94 GHz) is confirmed by experiment. Well-balanced outputs and a directivity of better than 40 dB are obtained. Though a substantial amount of insertion loss in the experimental model is found, this loss is believed to be largely dielectric loss. Design and performance data are presented.**

I. Introduction

IN THE PAST few years coupled-wave theory has become increasingly important due to its role in the millimeter-wave devices and integrated circuits. Coupled-wave theory between parallel transmission lines was originally introduced by Miller in a widely known paper [1]. Later, the problem of coupling between two parallel dielectric waveguides and its application to directional couplers has been studied extensively and numerous papers have been published [2]–[5]. Recently, the theory of coupling between parallel transmission lines has been extended to describe the coupling between nonparallel transmission lines [6]–[10]. Examples of practical interest are the application of coupling between two curved transmission lines to the design of directional couplers, and in some other cases, the need to avoid crosstalk [11], [12]. Although this case has received little attention, up to date, except for some recent work, such analysis is important due to the increasing interest in millimeter-wave integrated devices.

In a previous paper, [9] the authors have presented the basic formulation for the coupling between curved transmission lines, and considered the coupling problem of a finite length parallel coupler joined to matched loads via nonparallel transmission lines. Closed-form expressions for the field amplitudes, directivity, and reflection were also given. In the present paper, the field amplitude differential equations that were derived in the previous paper for the curved sections are used. By combining the WKB method and a perturbation technique, the differential equations are solved, and closed form expressions for the field amplitudes, the directivity, and the reflection coefficient (due to back-coupling) are derived. An experimental model for a 3-dB directional coupler designed at 94 GHz is constructed and tested. The experimental data are in good agreement

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